

A two-sided ideal trick in Hopf algebroid axiomatics

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Abstract

Recently S. Meljanac proposed a construction of a class of examples of an algebraic structure with properties very close to the Hopf algebroids H over a noncommutative base A of other authors. His examples come along with a subalgebra \mathcal{B} of $H \otimes H$, here called the balancing subalgebra, which contains the image of the coproduct and such that the intersection of \mathcal{B} with the kernel of the projection $H \otimes H \rightarrow H \otimes_A H$ is a two-sided ideal in \mathcal{B} which is moreover well behaved with respect to the antipode. We propose a set of abstract axioms covering this construction and make a detailed comparison to the Hopf algebroids of Lu. We prove that every scalar extension Hopf algebroid can be cast into this new set of axioms.

1 Introduction

Hopf algebroids [2, 5, 13] are generalizations of Hopf algebras, which are roughly in the same relation to groupoids as Hopf algebras are to groups. They are **bialgebroids** possessing a version of an antipode, where an (associative) bialgebroid is the appropriate generalization of a bialgebra. Hopf algebroids comprise several structure maps defined on a pair of associative unital algebras, the **total algebra** H (generalization of a function algebra on the space of morphisms of a groupoid), and the **base algebra** A (generalization of a function algebra on the space of objects (equivalently, units) of a groupoid). The main structure on the total algebra of a bialgebroid is an A -bimodule structure on H and the coproduct $\Delta : H \rightarrow H \otimes_A H$. The commutative Hopf algebroids (where both H and A are commutative) are easy to define by a categorical dualization of the groupoid concept. They are used as a classical tool in stable homotopy theory [8, 19]. The noncommutative Hopf algebroids over a commutative base (H noncommutative, A commutative) are also rather straightforward to introduce; this theory has been studied from late 1980-s, under the influence of the quantum group theory [17]. The most obvious examples are the convolution algebras of finite groupoids. Bialgebroids and Hopf algebroids over a noncommutative base are much more complicated to define; several nonequivalent versions were

developed since around 1992 by Schauenburg [20], Lu [13], Böhm [1, 2, 3, 4], Xu [26] and others. Many examples of Hopf algebroids over noncommutative bases have been studied in the contexts of inclusions of von Neumann algebra factors [2], dynamical Yang-Baxter equation, weak Hopf algebras, deformation quantization [26], noncommutative torsors, noncommutative differential calculus and cyclic homology [9, 10] etc.

In 2012, Meljanac devised a new approach to some examples of (topological) Hopf algebroids over a noncommutative base restricting the codomain of the coproduct map in a useful, but somewhat *ad hoc* way. To construct that codomain, he chooses a subalgebra \mathcal{B} in the tensor square $H \otimes H$ of the total algebra H , such that the intersection of \mathcal{B} with the kernel I_A of the projection $\pi : H \otimes H \rightarrow H \otimes_A H$ to the tensor square over the noncommutative base algebra A is a two-sided ideal $I_A \cap \mathcal{B} \subset \mathcal{B}$ (with an appropriate behaviour under the antipode map). The appearance of the two-sided ideal is a novel and somewhat unexpected feature reminding of the classical case where the base algebra A is commutative and I_A is two-sided itself. The approach is developed in collaborative works [12] with most details in [11].

These papers neglect two mathematical issues. Firstly, there is no care taken about the implicit use of completions: the values of the coproduct involve infinite sums, hence the codomain should be a completed tensor product. Secondly, at the algebraic level, they do not state a complete axiomatic framework for their version of Hopf algebroid, nor state its precise relation to other definitions. Instead they construct an interesting class of examples and give a partial list of essential properties. In our article [18] with Meljanac, we treat a somewhat wider class of examples in a mathematically rigorous way, using G. Böhm's definition of a symmetric Hopf algebroid, partly adapted to a formally completed tensor product. For a different adaptation which is an internal version to a category of filtered-cofiltered vector spaces (with a more sensible completion) see [23] and for the connections to formal geometry see [22]. These works took care of the first issue (completions), but went away from the original two-sided ideal approach. To address the second issue, we here propose a new set of axioms which expresses the essence of the two-sided ideal approach and discuss it in the context. The subalgebra $\mathcal{B} \subset H \otimes_k H$ in the new axioms is named the **balancing subalgebra** and the new version of the Hopf algebroid is named the Hopf algebroid with the balancing subalgebra \mathcal{B} .

This new version of Hopf algebroids over a noncommutative base, namely with a balancing subalgebra, is in Theorem 3.4 compared to the Hopf alge-

broids of Lu, because her version of the axioms for the antipode map involves a choice of a section which is close in spirit to the choice of balancing subalgebra in our axiomatics and in the original approach of Meljanac. Our main result is Theorem 4.12 (based on rather nontrivial Lemma 4.11) stating that every scalar extension Hopf algebroid can be cast into a Hopf algebroid with a (suitable choice of) balancing subalgebra \mathcal{B} .

2 Bialgebroids over noncommutative base

Bialgebroids and Hopf algebroids over a noncommutative base algebra are far less obvious to formulate [2, 6, 13, 20, 26]. To give it a try, recall that a \mathbf{k} -bialgebra is compatibly a monoid (algebra) and a comonoid (coalgebra) in the same monoidal category, namely that of \mathbf{k} -modules, where \mathbf{k} is the commutative ground ring. Thus, for a noncommutative base ring A , it is natural to try replacing the monoidal category of \mathbf{k} -modules with the monoidal category of A -bimodules. The latter is neither symmetric nor braided monoidal in general, so the usual compatibility condition between the comonoids and monoids makes no sense. Instead, it appears that the monoid and the comonoid part of the left A -bialgebroid structure live in different monoidal categories [2].

The monoid structure (H, μ, η) on H is in the monoidal category of $A \otimes A^{\text{op}}$ -bimodules; equivalently, (H, m) is an associative \mathbf{k} -algebra and the unit map η is a morphism of \mathbf{k} -algebras $\eta : A \otimes A^{\text{op}} \rightarrow H$ (we say that (H, μ, η) is an $A \otimes A^{\text{op}}$ -ring); map $\mu : H \otimes_{\mathbf{k}} H \rightarrow H$ satisfies $(h.a)k = h(a.k)$, $\forall a \in A$, $\forall h, k \in H$, hence it induces the map also denoted $\mu : H \otimes_A H \rightarrow H$. The unit η is usually described in terms of its left leg $\alpha := \eta(- \otimes 1_{A^{\text{op}}}) : A \rightarrow H$ and right leg $\beta := \eta(1_A \otimes -) : A^{\text{op}} \rightarrow H$, also called the source and target maps respectively; then, their images commute because $\alpha(a)\beta(b) = \eta(a \otimes b) = \beta(b)\alpha(a)$. An $A \otimes A^{\text{op}}$ -ring (H, μ, η) is below described as (H, μ, α, β) .

The comonoid structure of (H, Δ, ϵ) is instead in the monoidal category of A -bimodules (we say that H is an A -coring).

Definition 2.1. An $A \otimes A^{\text{op}}$ -ring (H, μ, α, β) and an A -coring (H, Δ, ϵ) on an A -bimodule H form a **left associative A -bialgebroid** $(H, \mu, \alpha, \beta, \Delta, \epsilon)$ if they satisfy the following compatibility conditions:

- (C1) the underlying A -bimodule structure of the A -coring structure is determined by the source map and target map (part of the $A \otimes A^{\text{op}}$ -ring

structure): $r.a.r' = \alpha(r)\beta(r')a$.

(C2) formula $\sum_{\lambda} h_{\lambda} \otimes f_{\lambda} \mapsto \epsilon(\sum_{\lambda} h_{\lambda} \alpha(f_{\lambda}))$ defines an action $H \otimes A \rightarrow A$ which extends the left regular action $A \otimes A \rightarrow A$ along the inclusion $A \otimes A \xrightarrow{\alpha \otimes A} H \otimes A$.

(C3) the linear map $h \otimes (g \otimes k) \mapsto \Delta(h)(g \otimes k)$, $H \otimes (H \otimes H) \rightarrow H \otimes H$, induces a well defined action $H \otimes (H \otimes_A H) \rightarrow H \otimes_A H$.

The condition (C1) implies that the kernel $I_A = \text{Ker } \pi$ of the projection map

$$\pi : H \otimes_{\mathbf{k}} H \rightarrow H \otimes_A H$$

of H -bimodules is a **right ideal** in the algebra $H \otimes_{\mathbf{k}} H$, generated by the set of elements of the form $\beta(a) \otimes 1 - 1 \otimes \alpha(a)$:

$$I_A = \{ \beta(a) \otimes_{\mathbf{k}} 1 - 1 \otimes_{\mathbf{k}} \alpha(a) \mid a \in A \} \cdot (H \otimes_{\mathbf{k}} H) \quad (1)$$

The third condition (C3) is here stated in the form of Lu. Let

$$H \times_A H = \left\{ \sum b_i \otimes b'_i \in H \otimes_A H \mid \sum_i b_i \otimes b'_i \alpha(a) = \sum_i b_i \beta(a) \otimes b'_i, \forall a \in A \right\},$$

which is an H -subbimodule of $H \otimes_A H$, called the Takeuchi product. Then $H \times_A H$ is, unlike $H \otimes_A H$, an associative algebra with respect to the componentwise product. This is not true for $H \otimes_A H$: the componentwise rule is not well defined because it may depend on the chosen representatives in $H \otimes_{\mathbf{k}} H$; this is because $I = I_A$ is only a *right* ideal in general. Ping Xu [26] has shown that (C3) is equivalent to the joint assertion of the following two requirements:

(C3a) $\text{Im } \Delta \subset H \times_A H$,

(C3b) Δ as a map from H to $H \times_A H$ is a homomorphism of algebras.

Of course, (C3b) makes sense only because of (C3a). A short calculation shows that $\pi^{-1}(H \times_A H)$ is also a subalgebra of $H \otimes H$. Observe a commutative diagram of A -bimodules:

$$\begin{array}{ccccc} & & \pi^{-1}(H \times_A H) & \longrightarrow & H \otimes H \\ & & \downarrow \pi|_{\pi^{-1}(H \times_A H)} & & \downarrow \pi \\ H & \xrightarrow{\Delta} & H \times_A H & \longrightarrow & H \otimes_A H \end{array} \quad (2)$$

All arrows except those into $H \otimes_A H$ are also homomorphisms of algebras.

The equation $\sum_i b_i \otimes b'_i \alpha(a) = \sum_i b_i \beta(a) \otimes b'_i$ for elements in $H \otimes_A H$ is demanded in the quotient, hence it holds only up to elements in I_A ; if we take the same equation strictly in $H \otimes H$ to cut some subalgebra (actually a left ideal) $H \tilde{\times} H \subset H \otimes H$, then the projection $\pi|_{H \tilde{\times} H}$ maps this subalgebra within $H \times_A H$, but is not necessarily onto. In the categorical language, $H \times_A H$ is an end (kind of a categorical limit) of a coend (kind of a colimit), not the other way around. However, Meljanac in his examples takes some other subalgebra $\mathcal{B} \subset H \otimes H$ (not a universal construction) first and then passes to the quotient by $\pi|_{\mathcal{B}}$ (hence a colimit), with a result which is still an algebra (different from $H \times_A H$). To achieve this, he needs that

(C3MI) $I_A \cap \mathcal{B}$ is a two-sided ideal in \mathcal{B} .

In addition, he (implicitly) requires

(C3Ma) $\text{Im } \Delta \subset \mathcal{B}/(I_A \cap \mathcal{B})$,

(C3Mb) Δ as a map from H to $\mathcal{B}/(I_A \cap \mathcal{B})$ is a homomorphism of algebras.

Definition 2.2. A left A -bialgebroid with a balancing subalgebra \mathcal{B} comprises an $A \otimes A^{\text{op}}$ -ring (H, m, α, β) and an A -coring (H, Δ, ϵ) on the same A -bimodule H which satisfy the compatibility conditions (C1) and (C2), and a (not necessarily unital) subalgebra $\mathcal{B} \subset H \otimes H$ satisfying (C3MI), (C3Ma) and (C3Mb) and called the **balancing subalgebra**.

A left A -bialgebroid with a balancing subalgebra \mathcal{B} is not necessarily a left associative A -bialgebroid in the standard sense, because (C3) does not always hold. However, if \mathcal{B} is the preimage $\pi^{-1}(H \times_A H)$ of the Takeuchi product under the natural projection π then (C3) follows. Conversely, given a left associative A -bialgebroid $H = (H, m, \alpha, \beta, \Delta, \epsilon)$, we call $\pi^{-1}(H \times_A H) \subset H \times_k H$ the **trivial balancing subalgebra** of H and it makes H into a left A -bialgebroid with the trivial balancing subalgebra.

Observe a commutative diagram of A -bimodules where all arrows except those into $H \otimes_A H$ are homomorphisms of algebras:

$$\begin{array}{ccccc}
 & \mathcal{B} & \longrightarrow & H \otimes H & \\
 & \downarrow \pi|_{\mathcal{B}} & & \downarrow \pi & \\
 H & \xrightarrow{\Delta} & \mathcal{B}/(I_A \cap \mathcal{B}) & \longrightarrow & H \otimes_A H
 \end{array} \tag{3}$$

Proposition 2.3. Let $(H, \mu, \alpha, \beta, \Delta, \epsilon)$ be the data defining an $A \otimes A^{\text{op}}$ -ring and A -coring satisfying (C1), (C2) and (C3a). Suppose there exist a subalgebra $\mathcal{B} \subset H \otimes H$ such that (C3MI) and (C3Ma) hold. Then these data define a A -bialgebroid with the balancing subalgebra \mathcal{B} iff they (without \mathcal{B}) define an associative A -bialgebroid; in other words, (C3b) holds iff (CM3b) holds.

Proof. This is a rather simple observation: (C3a) and (C3Ma) together imply that $\text{Im } \Delta \subset \mathcal{B}/(I_A \cap \mathcal{B}) \cap H \times_A H$ which has the structure of a subalgebra of $\mathcal{B}/(I_A \cap \mathcal{B})$ and also of $H \times_A H$; the algebra structures on $\mathcal{B}/(I_A \cap \mathcal{B})$ and $H \times_A H$ are both defined factorwise, hence equal on the intersection. Thus (C3b) and (C3Mb) hold iff $\Delta : H \rightarrow \mathcal{B}/(I_A \cap \mathcal{B}) \cap H \times_A H$ is an algebra map respectively with the same multiplication on the codomain, hence they are equivalent. \square

3 Hopf algebroids: antipode

Definition 3.1. A **Hopf A -algebroid** in the sense of J-L. Lu [13] (or a Lu-Hopf algebroid) is a left associative A -bialgebroid $(H, \mu, \alpha, \beta, \Delta, \epsilon)$ with an antipode map $\tau : H \rightarrow H$, which is a linear antiautomorphism satisfying

$$\tau\beta = \alpha \tag{4}$$

$$\mu(\text{id} \otimes_{\mathbf{k}} \tau)\gamma\Delta = \alpha\epsilon \tag{5}$$

$$\mu(\tau \otimes_A \text{id})\Delta = \beta\epsilon\tau \tag{6}$$

for some linear section $\gamma : H \otimes_A H \rightarrow H \otimes H$ of the projection $\pi : H \otimes H \rightarrow H \otimes_A H$.

The reason for introducing γ in (5) is the fact that $\mu(\text{id} \otimes_A \tau)\Delta$ is not a well defined map because $\mu(\text{id} \otimes_{\mathbf{k}} \tau)(I_A) \neq 0$ in general. Indeed, I_A is a linear span of the set of all elements of the form $\beta(a)h \otimes k - h \otimes \alpha(a)k$, where $a \in A$ and $h, k \in H$, and $\mu(\text{id} \otimes \tau)(\beta(a)h \otimes k - h \otimes \alpha(a)k) = \beta(a)h\tau(k) - h\tau(k)\tau(\alpha(a))$ which can be nonzero in general. No such problems occur with (6) because

$$\mu(\tau \otimes \text{id})(\beta(a)h \otimes k - h \otimes \alpha(a)k) = \tau(h)\tau(\beta(a))k - \tau(h)\alpha(a)k \stackrel{(4)}{=} 0.$$

Definition 3.2. A **Hopf A -algebroid with a balancing subalgebra \mathcal{B}** is a left A -bialgebroid $(H, \mu, \alpha, \beta, \Delta, \epsilon)$ with a balancing subalgebra \mathcal{B} together

with an algebra antihomomorphism $\tau : H \rightarrow H$, called the **antipode**, such that

$$\mu(\text{id} \otimes_{\mathbf{k}} \tau)(I_A \cap \mathcal{B}) = 0 \quad (7)$$

$$\tau\beta = \alpha \quad (8)$$

$$\mu(\text{id} \otimes_A \tau)\Delta = \alpha\epsilon \quad (9)$$

$$\mu(\tau \otimes_A \text{id})\Delta = \beta\epsilon\tau \quad (10)$$

Two equations are the same as before: (8) is identical to (4) and (10) to (6). Equation (9) makes sense because of (7). Notice that there is no need for a choice of a section γ . Choice of the subalgebra \mathcal{B} is intuitively a wider choice which accomplishes the same.

Remark 3.3. The map $\mu(\text{id} \otimes_{\mathbf{k}} \tau) : h \otimes h' \rightarrow h\tau(h')$ is linear but neither a homomorphism nor an antihomomorphism of algebras; hence it is not sufficient to check (7) on the algebra generators (or even worse, on ideal generators) of $I_A \cap \mathcal{B}$. This will be the central difficulty in Section 4.

Theorem 3.4. If a Hopf algebroid with a balancing subalgebra satisfies (C3a) then it admits a structure of a Lu-Hopf algebroid.

Proof. Choose a vector space splitting of $H \otimes_A H$ into $\text{Im } \Delta$ and the linear complement; for γ take any linear section of the projection $\pi : H \otimes H \rightarrow H \otimes_A H$ such that values $\gamma(p)$ over points $p \in \text{Im } \Delta$ are in \mathcal{B} (this can be done by (C3Ma)) and on the linear complement prescribe any linear choice for γ , for instance 0. Condition (C3b) holds by (CM3b) and Proposition 2.3. Then $\mu(\text{id} \otimes_{\mathbf{k}} \tau)\gamma\Delta(h) = \mu(\text{id} \otimes_A \tau)\Delta(h)$ as the right hand side is defined by choosing any representative of $\Delta(h)$ in $H \otimes_{\mathbf{k}} H$ and evaluating $\mu(\text{id} \otimes_{\mathbf{k}} \tau)$. Thus (5) holds, and the other conditions on the antipode are identities. \square

We do not expect the converse of Theorem 3.4 in general, but we show below that every scalar extension is in both classes. Clearly, every Hopf algebroid over a commutative base is also in both classes.

4 Scalar extension Hopf algebroids

Let T be a Hopf \mathbf{k} -algebra with comultiplication $\Delta_T : T \rightarrow T \otimes_{\mathbf{k}} T$ and antipode S . Let A be a braided-commutative left-right Yetter-Drinfeld T -module algebra. That means that A is a unital associative algebra with a left

T -action $\blacktriangleright: T \otimes A \rightarrow A$ which is Hopf ($t \blacktriangleright (ab) = (t_{(1)} \blacktriangleright a)(t_{(2)} \blacktriangleright b)$) and a right T -coaction $a \mapsto a_{[0]} \otimes a_{[1]}$ which is morphism of algebras $A \rightarrow A \otimes T^{\text{op}}$ (see [5]), satisfying the left-right Yetter-Drinfeld condition

$$(t_{(1)} \blacktriangleright a_{[0]}) \otimes (t_{(2)} a_{[1]}) = (t_{(2)} \blacktriangleright a)_{[0]} \otimes (t_{(2)} \blacktriangleright a)_{[1]} t_{(1)}, \quad \forall t \in T, \forall a \in A.$$

and the braided commutativity $x_{[0]}(x_{[1]} \blacktriangleright a) = ax$ for all $a, x \in A$.

Lemma 4.1. Braided commutativity condition is equivalent to the condition $((Sd_{[1]} \blacktriangleright a)d_{[0]} = da$ for all $d, a \in A$.

Proof. This is rather standard. We show for convenience one direction: assuming braided commutativity,

$$\begin{aligned} da &= d_{[0]}((d_{[1]}Sd_{[2]}) \blacktriangleright a) \\ &= d_{[0][0]}(d_{[0][1]} \blacktriangleright ((Sd_{[1]}) \blacktriangleright a)) \\ &= ((Sd_{[1]}) \blacktriangleright a)d_{[0]} \end{aligned}$$

The proof of another direction is similar. \square

The smash product $H = A \sharp T$ can be equipped with the structure of the total algebra of a Hopf A -algebroid (which is below called scalar extension Hopf algebroid), either in sense of Lu [13], but also in the sense of the symmetric Hopf algebroid of Böhm. The main formulas are

$$\alpha(a) = a \sharp 1, \quad \beta(a) = a_{[0]} \sharp a_{[1]}, \quad \Delta(a \sharp t) = (a \sharp t_{(1)}) \otimes_A (1 \sharp t_{(2)}).$$

We often identify $A \sharp 1 = \text{Im } \alpha$ with A and $1 \sharp T$ with T . By the Definition 1 the right ideal $I_A \subset H \otimes_A H$ is generated by the set of all elements of the form

$$I(a) = \beta(a) \otimes 1 - 1 \otimes \alpha(a) = a_{[0]} \sharp a_{[1]} \otimes 1 - 1 \otimes a, \quad a \in A.$$

There is also another set of generators $R(a)$ of I_A , more convenient for our analysis below.

Proposition 4.2. In the case of scalar extension $H = A \sharp T$, I_A is generated as a right ideal of $H \otimes_{\mathbf{k}} H$ by the set of all elements of the form

$$R(a) = a \otimes 1 - Sa_{[1]} \otimes a_{[0]}, \quad a \in A. \tag{11}$$

Proof. In the notation introduced before the proposition,

$$I(a) = (a_{[0]} \sharp 1 - Sa_{[0][1]} \otimes a_{[0][0]})(a_{[1]} \otimes 1) = R(a_{[0]})(a_{[1]} \otimes 1).$$

Notice that $a_{[0]} \in A$. On the other hand,

$$R(a) = (a_{[0][0]} \sharp a_{[0][1]} \otimes 1 - 1 \otimes a_{[0]})(Sa_{[1]} \otimes 1) = I(a_{[0]})(Sa_{[1]} \otimes 1).$$

Therefore, the right ideal generated by the sets of all $I(a)$ and the right ideal generated by all $R(a)$ coincide. \square

We notice that the elements R_μ in [11] are $R_\mu = R(\hat{x}_\mu)$ where \hat{x}_μ , $\mu = 1, \dots, \dim \mathfrak{g}$ are the generators of the universal enveloping algebra $U(\mathfrak{g})$ corresponding to the linear basis of the Lie algebra \mathfrak{g} in their work. See 4.3.

Lu's section. For any scalar extension bialgebroid, J-H. Lu [13] introduces a section $\gamma : H \otimes_A H \rightarrow H \otimes_{\mathbf{k}} H$ as the unique \mathbf{k} -linear extension of the formula

$$\gamma : h \otimes_A (a \sharp t) \mapsto \beta(a)h \otimes (1 \sharp t), \quad (12)$$

where $h \in H$, $a \in A$ and $t \in T$. This formula defines a section of the projection $H \otimes_{\mathbf{k}} H \rightarrow H \otimes_A H$ because $a \sharp t = \alpha(a)(1 \sharp t)$; hence $h \otimes_A (a \sharp t) = \beta(a)h \otimes_A (1 \sharp t)$. Section γ is well defined by the above formula, as on the elements of the form $\beta(b)h \otimes (c \sharp \tilde{t}) - h \otimes \alpha(b)(c \sharp \tilde{t})$ the formula gives $\beta(b)\beta(c)h \otimes (1 \sharp \tilde{t}) - \beta(bc)h \otimes (1 \sharp \tilde{t}) = 0$.

In particular, the formula (12) gives

$$(\gamma \circ \Delta)(a \sharp t) = (a \sharp t_{(1)}) \otimes (1 \sharp t_{(2)}). \quad (13)$$

4.1 Subalgebra $W \subset H \otimes H$ where $H = A \sharp T$ is a scalar extension Hopf algebroid over A

Notation 4.3. Let T be a Hopf algebra and A a braided commutative algebra in the category of left-right Yetter-Drinfeld T -modules. For a scalar extension $A \sharp T$ let $W \subset (A \sharp T) \otimes (A \sharp T)$ be the smallest unital subalgebra such that all elements of the form $a \otimes 1$ and all elements of the form $Sa_{[1]} \otimes a_{[0]}$ (where $a \in A \cong A \sharp 1 \subset A \sharp T$) are in W . Let W^+ be the two sided ideal in W generated by all elements of the form $R(a) = a \otimes 1 - Sa_{[1]} \otimes a_{[0]}$ where $a \in A$ (compare (11)).

Let $W_0^+ \subset W$ be the linear subspace of W spanned by all elements of the form $(x \otimes 1 - Sx_{[1]} \otimes x_{[0]})(x' \otimes 1)$ where $x, x' \in A \cong A\sharp 1$.

We formulate Lemma 1 and Lemma 2 which together imply $W_0^+ = W^+$.

Lemma 4.4. For $x, z \in A$ we have $(x \otimes 1)(z \otimes 1 - Sz_{[1]} \otimes z_{[0]}) \in W_0^+$.

Proof. Multiplying out, and using $xs(t) = S(t_{(1)})t_{(2)}xS(t_{(3)}) = S(t_{(1)})(t_{(2)} \blacktriangleright x)$ for $x \in A, t \in T$, we obtain

$$\begin{aligned} xz \otimes 1 - xS(z_{[1]}) \otimes z_{[0]} &= xz \otimes 1 - S(z_{[1]})(z_{[2]} \blacktriangleright x) \otimes z_{[0]} \\ &= \text{by braided commutativity} \\ &= z_{[0]}(z_{[1]} \blacktriangleright x) \otimes 1 - S(z_{[1]})(z_{[2]} \blacktriangleright x) \otimes z_{[0]} \\ &= (z_{[0]} \otimes 1 - S(z_{[0][1]})) \otimes z_{[0][0]}(z_{[2]} \blacktriangleright x \otimes 1) \end{aligned}$$

and the right hand side is clearly in W_0^+ as claimed. \square

Lemma 4.5. $R(x)R(z) = (x \otimes 1 - Sx_{[1]} \otimes x_{[0]})(z \otimes 1 - Sz_{[1]} \otimes z_{[0]}) \in W_0^+$.

Proof. Since $x \mapsto x_{[1]} \otimes x_{[0]}$ is a morphism of algebras $A \rightarrow T^{\text{op}} \otimes_{\mathbf{k}} A$ and $T \otimes_{\mathbf{k}} A \hookrightarrow A\sharp T \otimes_{\mathbf{k}} A\sharp T = H \otimes_{\mathbf{k}} H$ inclusion of algebras, we conclude that $x \mapsto Sx_{[1]} \otimes x_{[0]}$ is a morphism of algebras $A \rightarrow H \otimes_{\mathbf{k}} H$ (with respect to the componentwise multiplication in $H \otimes_{\mathbf{k}} H$). Therefore,

$$\begin{aligned} &(x \otimes 1 - Sx_{[1]} \otimes x_{[0]})(z \otimes 1 - Sz_{[1]} \otimes z_{[0]}) = \\ &= (x \otimes 1 - Sx_{[1]} \otimes x_{[0]})(z \otimes 1) + xSz_{[1]} \otimes z_{[0]} - xz \otimes 1 + xz \otimes 1 - S(xz)_{[1]} \otimes (xz)_{[0]} \\ &= (x \otimes 1 - Sx_{[1]} \otimes x_{[0]})(z \otimes 1) + (-x \otimes 1)(z \otimes 1 - Sz_{[1]} \otimes z_{[0]}) + (xz \otimes 1 - S(xz)_{[1]} \otimes (xz)_{[0]}). \end{aligned}$$

The first and the third summands on the right hand side are manifestly in W_0^+ while for the second summand we apply Lemma 4.4. \square

Corollary 4.6. (i) $(Sx_{[1]} \otimes x_{[0]})(z \otimes 1 - Sz_{[1]} \otimes z_{[0]}) \in W_0^+$,

(ii) W_0^+ is a two-sided ideal in W ,

(iii) $W^+ = W_0^+$.

Proof. (i) follows by subtracting the expression in the statement of Lemma 2 from the expression in the statement of Lemma 1.

(ii) W_0^+ is a right ideal: it is obvious that we can multiply by $z \otimes 1$ from the right; this together with the assertion of Lemma 4.5 implies that we can also multiply by $Sz_{[1]} \otimes z_{[0]}$ from the right.

(ii) W_0^+ is a left ideal: using Lemma 4.4, $(x \otimes 1)R(z)(x' \otimes 1) \in W_0^+(x' \otimes 1)$ what is in W_0^+ as it is a right ideal. Combining with Lemma 4.5 we also get $(Sx_{[1]} \otimes x_{[0]})R(z)(x \otimes 1) \in W_0^+$.

For (iii) notice first that trivially $W_0^+ \subset W^+$. For the converse inclusion, $W^+ \subset W_0^+$, it is sufficient to observe that $R(a) \in W_0^+$, apply (ii) and the definition of W^+ . \square

Let now τ be the antipode of the scalar extension Hopf algebroid $A\sharp T$ over A . We know (cf. [6, 13]) that

$$\tau(a\sharp t) = S(t)S^2(a_{[1]}) \cdot a_{[0]}. \quad (14)$$

Theorem 4.7. $\mu(id \otimes_{\mathbf{k}} \tau)W^+ = \{0\}$.

Proof. By Corollary (iii) $W^+ = W_0^+$, which is the span of the elements of the form

$$(x \otimes 1 - Sx_{[1]} \otimes x_{[0]})(z \otimes 1), \quad \text{where } x, z \in A.$$

We can easily compute $\mu(id \otimes \tau)$ on such an element as

$$xz - S(x_{[2]})zS^2(x_{[1]})x_{[0]} = xz - ((Sx_{[1]}) \blacktriangleright z)x_{[0]} = 0.$$

by the braided commutativity. \square

4.2 Subalgebra \mathcal{B} and two-sided ideal $\mathcal{B}^+ \subset \mathcal{B}$

In this section, we want to show that every scalar extension Lu-Hopf algebroid $H = A\sharp T$ is also a Hopf algebroid with a (carefully chosen) balancing subalgebra \mathcal{B} .

Using the inclusion $T \otimes_{\mathbf{k}} T \hookrightarrow A\sharp T \otimes_{\mathbf{k}} A\sharp T$, we identify the image of the coproduct $\Delta_T : T \rightarrow T \otimes_{\mathbf{k}} T$ of the Hopf algebra T with a subalgebra of $H \otimes_{\mathbf{k}} H$ which will be denoted by $\Delta_T(T)$.

Definition 4.8. Let the subalgebra $\mathcal{B} \subset A\sharp T \otimes_{\mathbf{k}} A\sharp T$ be the subalgebra generated by W and $\Delta_T(T)$ or, equivalently, by the set

$$\{X \otimes 1, SX_{[1]} \otimes X_{[0]} \mid X \in A\} \cup \Delta_T(T).$$

The elements of this set will be called the **distinguished generators of \mathcal{B}** .

Let \mathcal{B}^+ be the two-sided ideal in \mathcal{B} generated by the subset

$$\{X \otimes 1 - SX_{[1]} \otimes X_{[0]} \mid X \in A\} \subset \mathcal{B},$$

whose elements are called the **distinguished generators of \mathcal{B}^+** .

Theorem 4.9. (i) $\mathcal{B}^+ = I_A \cap \mathcal{B}$.

(ii) (CRMI) holds: $I_A \cap \mathcal{B}$ is a two-sided ideal in \mathcal{B} .

(iii) (CRMa) holds: $\text{Im } \Delta \subset \mathcal{B}/(I_A \cap \mathcal{B})$

(iv) The scalar extension is a bialgebroid with the balancing subalgebra \mathcal{B} .

(v) The algebra $\mathcal{B}/\mathcal{B}^+$ is isomorphic to $\Delta_L(A\sharp T) \subset A\sharp T \otimes_A A\sharp T$.

Proof. (i) follows immediately by the Definition 4.8 for \mathcal{B}^+ and Proposition 4.2.

(ii) and (iii) is immediate check knowing the generators and (i).

For (iv) use (ii), (iii), Proposition 2.3 and the fact that the (C3b) is known for scalar extensions [5, 13]. \square

Remark 4.10. For the simplicity of exposition, in the following multilinear calculations we omit the summation signs and the corresponding labels when generically denoting simple products of distinguished generators of \mathcal{B} : we use Sweedler-like symbols. The generic symbol $K_i \otimes L_i$ for a distinguished generator in the following lemma and theorem is a symbol for a sum of the form

$$\sum_{\alpha_i} K_i^{\alpha_i} \otimes L_i^{\alpha_i},$$

whose components $K_i^{\alpha_i} \otimes L_i^{\alpha_i}$ are equal to components of one of these sums: $X \otimes 1$, $SX_{[1]} \otimes X_{[0]}$, $f_{(0)} \otimes f_{(1)}$, where $X \in A$ and $f \in T$. When we say K_i is in $A\sharp 1$, it means that each $K_i^{\alpha_i} \in A\sharp 1$, because by this notation K_i denotes each $K_i^{\alpha_i}$ at the same time. Similarly, the generic symbol $K \otimes L$ for a product of distinguished generators $K_i \otimes L_i$,

$$\prod_{i=1}^n \sum_{\alpha_i} (K_i^{\alpha_i} \otimes L_i^{\alpha_i}) = \sum_{\alpha_1, \dots, \alpha_n} \prod_{i=1}^n (K_i^{\alpha_i} \otimes L_i^{\alpha_i}) = \sum_{\alpha_1, \dots, \alpha_n} \prod_{i=1}^n K_i^{\alpha_i} \otimes \prod_{i=1}^n L_i^{\alpha_i},$$

is a symbol for a sum

$$\sum_{\alpha_1, \dots, \alpha_n} \prod_{i=1}^n K_i^{\alpha_i} \otimes \prod_{i=1}^n L_i^{\alpha_i}$$

and K and L are generic symbols denoting each product of components $\prod_{i=1}^n K_i^{\alpha_i}$ and $\prod_{i=1}^n L_i^{\alpha_i}$ at the same time.

Lemma 4.11. Let U be a product of distinguished generators of \mathcal{B} . Then

$$\mu(\text{id} \otimes \tau)(U) \in A\sharp 1.$$

Proof. Denote the distinguished generators that U is a product of by $K_1 \otimes L_1, \dots, K_n \otimes L_n$. The antipode τ is an algebra antihomomorphism, hence

$$\mu(\text{id} \otimes \tau)(U) = K_1 K_2 \cdots K_n \tau(L_n) \cdots \tau(L_1). \quad (15)$$

We claim that $K_i \cdot (Z\sharp 1) \cdot \tau(L_i) \in A\sharp 1$ for $Z\sharp 1 \in A\sharp 1$. We inspect this claim case by case as follows.

(a) If $K_i \otimes L_i \in \{X \otimes 1, SX_{[1]} \otimes X_{[0]} \mid X \in A\}$, then by (14)

$$K_i \cdot (Z\sharp 1) \cdot \tau(L_i) = K_i \cdot Z\sharp 1 \cdot S^2(L_{i[1]}) \cdot L_{i[0]}. \quad (16)$$

The product \cdot is in the smash product $A\sharp T$; A is identified there with $A\sharp 1$ and T with $1\sharp T$. There are now two subcases, (a1) and (a2).

(a1) For $K_i \otimes L_i = X \otimes 1$, (16) equals $XZ = XZ\sharp 1$ which is in $A\sharp 1$.

(a2) For $K_i \otimes L_i = SX_{[1]} \otimes X_{[0]}$, (16) equals $SX_{[2]} \cdot Z \cdot S^2(X_{[1]}) \cdot X_{[0]} = ((SX)_{[1]} \blacktriangleright Z) \cdot (SX)_{[2]} S((SX)_{[3]}) \cdot X_{[0]} = ((SX)_{[1]} \blacktriangleright Z) \cdot X_{[0]} = XZ$ which is in $A\sharp 1$.

(b) If $K_i \otimes L_i = f_{(1)} \otimes f_{(2)}$, then $K_i \cdot Z\sharp 1 \cdot \tau(L_i) = f_{(1)} \cdot Z\sharp 1 \cdot S(f_{(2)}) = (f \blacktriangleright Z)\sharp 1$ which is again in $A\sharp 1$.

Therefore the claim $K_i \cdot Z\sharp 1 \cdot \tau(L_i) \in A\sharp 1$ follows. In particular, for $Z = 1$, $K_n \tau(L_n) \in A\sharp 1$ so the claim provides the step and the base for the induction on p , where $0 \leq p \leq n - 1$, proving that

$$K_{n-p} \cdots K_n \tau(L_n) \cdots \tau(L_{n-p}) \in A\sharp 1.$$

By (15) it follows that $\mu(\text{id} \otimes \tau)(U) \in A\sharp 1$; in other words $\mu(\text{id} \otimes \tau)(U)$ is of the form $D\sharp 1$ where $D \in A$. \square

Theorem 4.12. Let $\tau : A\sharp T \rightarrow A\sharp T$ given by the formula (14) be the antipode of the scalar extension as a Lu-Hopf algebroid. Then

- (i) $\mu(\text{id} \otimes \tau)\mathcal{B}^+ = \{0\}$.
- (ii) τ makes the corresponding A -bialgebroid with a balancing subalgebra from Theorem 4.12 into a Hopf A -algebroid with a balancing subalgebra.

Proof. (i) A general element of \mathcal{B}^+ is a linear combination of the elements of the form

$$\prod_j M_j \otimes N_j \cdot (X \otimes 1 - SX_{[1]} \otimes X_{[0]}) \cdot \prod_k K_k \otimes L_k,$$

where $M_j \otimes N_j$, $K_k \otimes L_k$ are generic symbols for some distinguished generators of \mathcal{B} , and the middle factor is some distinguished generator in \mathcal{B}^+ . Notice that M_j , N_j , K_k , $L_k \in A\sharp 1 \cup 1\sharp T$, $X \in A$. Here M_j is a generic symbol denoting each M_j^α at the same time, see Remark 4.10.

By the linearity of $\mu(\text{id} \otimes \tau)$ it is sufficient to prove the assertion for one element of the form above. We decompose it in two pieces, multiply component by component and denote so obtained products by generic symbols M , N , K and $L \in A\sharp T$.

$$(M \cdot X\sharp 1 \cdot K) \otimes (N \cdot L) - (M \cdot 1\sharp SX_{[1]} \cdot K) \otimes (N \cdot X_{[0]}\sharp 1 \cdot L)$$

We apply map $\mu(\text{id} \otimes \tau)$ to this expression. Thus we need to show that

$$M \cdot X\sharp 1 \cdot K \cdot \tau(L) \cdot \tau(N) = M \cdot 1\sharp SX_{[1]} \cdot K \cdot \tau(L) \cdot \tau(X_{[0]}\sharp 1) \cdot \tau(N).$$

This directly follows by a simpler formula:

$$X\sharp 1 \cdot E = 1\sharp SX_{[1]} \cdot E \cdot \tau(X_{[0]}\sharp 1),$$

where E denoted $K \cdot \tau(L) \in A\sharp T$.

By Lemma 4.11, $K \cdot \tau(L) \in A\sharp 1$.

Therefore to finish the proof of the assertion (i) it is sufficient to show that for all X , $D \in A$ we have

$$X\sharp 1 \cdot D\sharp 1 = 1\sharp SX_{[1]} \cdot D\sharp 1 \cdot \tau(X_{[0]}\sharp 1),$$

where $\tau(X_{[0]}\sharp 1) = \alpha_R(X_{[0]}) = S^2(X_{[1]}) \cdot X_{[0]}$.

This amounts to show

$$XD\sharp 1 = ((SX)_{[1]} \blacktriangleright D) \sharp (SX)_{[2]} S((SX)_{[3]}) \cdot X_{[0]}\sharp 1,$$

that is,

$$XD\sharp 1 = ((SX)_{[1]} \blacktriangleright D) X_{[0]}\sharp 1,$$

which is equivalent to the braided commutativity for $A\sharp T$. Therefore, (i) is proven.

For the part (ii), according to Theorem 4.9, part (iv) it remains only to check the axioms for the antipode. The antipode requirements (8) and (10) have the same meaning as in the case of Lu-Hopf definition hence they are true. Now, thanks to (ii) the right-hand side of the equation (10), that is, $\mu(\text{id} \otimes \tau)\Delta = \text{id}$, does not depend on the representatives of $\Delta(at) = (a\sharp t_{(1)}) \otimes (1\sharp t_{(2)})$ in $H \otimes_{\mathbf{k}} H$ where $a \in A$ and $t \in T$. So we should compute that

$$(a\sharp t_{(1)})(S^2((at_{(2)})_{[1]}) \cdot (at_{(2)})_{[0]}) = a\sharp t,$$

which is the same computation as the one for Lu's choice of γ , see (13). Our result is stronger only in the sense that the additional freedom in \mathcal{B}_+ is allowed here and that \mathcal{B} is a balancing subalgebra in the sense of bialgebroids. \square

4.3 Comparison with the examples of Meljanac

Meljanac has devised his method [11, 12] to study topological Hopf algebroids related to a Lie algebra \mathfrak{g}_κ with the universal enveloping algebra $U(\mathfrak{g}_\kappa)$ in physics literature called the κ -Minkowski space. The same Hopf algebroid from the point of view of Lu-Hopf algebroid (and also in physics language) has been studied in [14] and (in just slightly more mathematical treatment) in [15]. Works [11, 12] made it clear that their construction applies to any finite dimensional Lie algebra \mathfrak{g} in characteristic zero. We comment below on how our construction of \mathcal{B} relates to theirs in this special case. As stated in the introduction, we neglect here the issues related to the adaptation of the notion of Hopf algebroid to the completed tensor products [18].

We use the notation from [18]. Generators of the Lie algebra \mathfrak{g} are denoted $\hat{x}_1, \dots, \hat{x}_n$ with commutators $[\hat{x}_\mu, \hat{x}_\nu] = C_{\mu\nu}^\lambda \hat{x}_\lambda$ and the generators of the symmetric algebra of the dual $S(\mathfrak{g}^*)$ by $\partial^1, \dots, \partial^n$. The completed dual $T = \hat{S}(\mathfrak{g}^*)$ is a topological Hopf algebra, namely the coproduct $\Delta_T : \hat{S}(\mathfrak{g}^*) \rightarrow \hat{S}(\mathfrak{g}^*) \hat{\otimes} \hat{S}(\mathfrak{g}^*)$ may be identified (using the symmetrization map $S(\mathfrak{g}) \cong U(\mathfrak{g})$ which is an isomorphism of coalgebras [18] and its dual isomorphism of algebras $\hat{S}(\mathfrak{g}^*) \cong U(\mathfrak{g}^*)$) the transpose (dual) map to the multiplication

$U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Now $A = U(\mathfrak{g})$ becomes a braided-commutative Yetter-Drinfeld module algebra over T ; this is related to a right Hopf action of $U(\mathfrak{g})$ on T which helps building the smash product $H = T \sharp A$. Regarding that the coproduct of $U(\mathfrak{g})$ is cocommutative, one can take instead a left Hopf action (composing the right action with the antipode) of $U(\mathfrak{g})$ on T and the smash product $A \sharp T$, what is the situation in this article and in [18].

If $R(a)$ is defined by (11), then one observes that $R(\hat{x}_\mu)$ equals R_μ of [11, 12]). It is known that $[R_\mu, R_\nu] = C_{\mu\nu}^\lambda R_\lambda$ (Eq. (32) in [11]). Moreover, there are no other relations among R_μ , hence the subalgebra in $H \otimes_{\mathbf{k}} H$ generated by R_μ is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$, but with generators R_μ in place of \hat{x}_μ ; thus denote it by $U(R)$. The relation [11] (33) $[\hat{x}_\mu \otimes 1, R_\nu] = C_{\mu\nu}^\lambda R_\lambda$ shows that the products of elements in $U(R)$ with elements in $A \sharp 1 \otimes_{\mathbf{k}} \mathbf{k} \subset A \sharp T \otimes_{\mathbf{k}} A \sharp T$ span a subalgebra in $H \otimes_{\mathbf{k}} H$. This is precisely our subalgebra W in this case. However, their relations (3.1), (3.2) in [11], which they used to prove that it is a subalgebra of $H \otimes_{\mathbf{k}} H$, do not have a simple analogue for general scalar extensions. It is also not clear what is the exact structure of the subalgebra generated by $R(a)$ -s in general. On the other hand, our Hopf algebraic definition (11) of $R(a)$ and the corresponding definition of W in the subsection 4.1 along with the lemmas therein guarantee that such general W is still a subalgebra in $H \otimes_{\mathbf{k}} H$.

The issues are substantially more complicated when we pass from W to \mathcal{B} . In the enveloping algebra case, the subalgebra \mathcal{B} (denoted $\hat{\mathcal{B}}$ in [11]) is defined in [11] rather easily as the subalgebra of all elements of the form $\sum_i w_i \Delta_{\hat{S}(\mathfrak{g})}(t_i)$ where $w_i \in W$, $t_i \in T = \hat{S}(\mathfrak{g}^*)$ are arbitrary (the sums may be infinite, in an appropriate completion). Equations (3.1)-(3.4) in [11] for general Lie algebra case, give $[\Delta \partial^\mu, R_\nu] = 0$ and $[\hat{x}_\mu \otimes 1, \Delta(\partial_\nu)] \in \Delta_T(T)$ and imply immediately that these elements form a subalgebra in $H \otimes_{\mathbf{k}} H$, and that \mathcal{B} has a very simple structure of all sums of products of the form: an element in $U(R)$ times an element in $A \sharp 1 \otimes_{\mathbf{k}} \mathbf{k} \subset H \otimes_{\mathbf{k}} H$ times an element of the form $\Delta_T(t_i)$ with $t_i \in T$. Similar structure as sum of products in three subalgebras in this fixed order for general scalar extension Hopf algebroids: in that case P_i does not commute with elements in W and multiple products (e.g. of the form $wtw't'w''$) of elements in W and elements in $\Delta_T(T)$ may appear, as analysed in the subsection 4.2. Regarding that $\mu(id \otimes_{\mathbf{k}} \tau)$ is not an antihomomorphism, the multiple products bring the main difficulty in our proof that the antipode τ is well defined (see Theorem 4.12 (i)).

Analogous comparisons may be made for the ideal $\mathcal{B}_+ = I_A \cap \mathcal{B}$ which is in [11] not defined as the intersection, but rather constructed exactly as \mathcal{B}

but with the enveloping algebra $U(R)$ replaced by its ideal $U_+(R) \subset U(R)$ of elements which are not degree 0 in the standard filtration of the universal enveloping algebra. The commutation relations (3.1)-(3.4) in [11] imply that such \mathcal{B}_+ is indeed a two-sided ideal in \mathcal{B} .

Our approach also differs from [11] in insisting that the coproduct is still defined as taking values in $H \otimes_A H$; the two-sided ideal trick is used only to make sense of the requirement and to check that the induced map into $\mathcal{B}/(I_A \cap \mathcal{B})$ is a morphism of algebras. Moreover they view \mathcal{B} as an abstract algebra constructed from its pieces $U(R)$, $A\sharp 1 \otimes_{\mathbf{k}} \mathbf{k}$ and $\Delta_T(T) \cong T$. In our approach, the coherently associative tensor product of bimodules \otimes_A is used to formulate the coassociativity of the coproduct as in Lu's, Xu's and Böhm's definitions. In [11] the coproduct is taking values in $\mathcal{B}/(I_A \cap \mathcal{B})$ by the definition and, in the spirit of their viewpoint, the higher analogues of $\mathcal{B}^{(k)} \subset H \otimes_{\mathbf{k}} H \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} H$ of $\mathcal{B} \subset H \otimes_{\mathbf{k}} H$ and the higher analogues $\mathcal{B}_+^{(k)}$ of $I_A \cap \mathcal{B}$ are constructed in order to deal with the (co)associativity issues. These are interesting structures but in our view more cumbersome than the familiar usage of the bimodule tensor product \otimes_A .

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